

ERRONEOUS EXAMPLES AS A SOURCE OF LEARNING IN MATHEMATICS

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ABSTRACT

We analyze why and how erroneous examples can be beneficially employed in learning mathematics. The 'Why' addresses reasoning and attitudes that are rarely fostered in today's mathematics lessons but which are useful for learning, certainly needed in real life and in all applications of mathematics as well as in maths itself. The 'How' addresses pedagogical strategies capitalizing on errors via erroneous examples and their realization in the web-based learning environment **ActiveMath**. An analysis of classes of errors informs the knowledge representation, the instructional strategies and the feedback of **ActiveMath**.

KEYWORDS

cognition in education, meta-cognition

1. MOTIVATION

New visions for school mathematics have been articulated in a number of papers and reports, e.g., [12, 13]. They all suggest a shift from routine and factual knowledge only to more emphasis on developing competencies such as solving maths-related problems, reasoning, and communicating mathematically. Consequently, an e-Learning environment should offer opportunities stimulating deep reasoning rather than shallow learning only and

- encourage students to explore
- show students that many mathematical questions have more than one correct answer
- teach the importance of careful reasoning and understanding
- build confidence in all students that they can learn mathematics
- help students verbalize their ideas.

While exploiting, verbalizing, looking for alternatives and trying to truly understand the maths mistakes occur naturally and are an essential part of learning. This has been widely neglected in traditional mathematics education. The behaviourist view of learning that informs much of traditional schooling is not likely to invite students and teachers to see errors in a positive light. Behaviourism assumes that learning is enhanced, when correct responses are rewarded (positive reinforcement) and incorrect ones are either punished or extinguished through lack of attention (withholding of positive reinforcement) [11]. Within this framework, paying explicit attention to (mathematical) errors in class is even considered by many as dangerous since it could interfere with fixing the correct result in the student's mind. Hence, traditionally, schools mostly teach 'positive knowledge' only and 'negative knowledge' is mostly avoided. Approaches to use errors as learning

opportunities can help to overcome the traditional transmission view of mathematics teaching and learning.

As opposed to mathematics, the value of engaging students in the analysis of their errors has been recognized in computer programming and language learning. In programming courses it is an established practice to expect students to debug their incorrect programs. The process of figuring out what went wrong may help even beginners to better understand the programming language. Similarly, in language training, learning from mistakes is a common practice, whereas in science and specifically in mathematics it is not [14].

Based on their empirical investigations in schools, Oser and Hascher [14] call for an error-culture at schools in which making mistakes is accepted and mistakes have to be analyzed together with a learner in order to let him find corrections. They analyzed that even if teacher want and are able to develop an error-culture in classrooms, they do not have the time for such an individual treatment of learners. Why not support this teaching strategy with a system then?

Therefore, a goal of the research for the web-based learning environment **ActiveMath** [10] is to develop tools, feedback and exercises that stimulate detecting, reasoning about, and correcting errors. One of the preconditions for this is the development of an appropriate knowledge representation that allows to identify erroneous parts and their correct counterparts. Moreover, adaptive course generation and feedback are key steps in this research plan.

This paper is the first in a row of research results. It concentrates on analyzing the learning opportunities of erroneous examples and suggest how to use them in an adaptive learning environment. This analysis informs the knowledge representation and the instructional strategies that we are developing in **ActiveMath**.

2. ERRONEOUS EXAMPLES

Worked examples, i.e., problems with (multi-step) solutions, have been a focus of psychological and ITS-research for a while. Controlled experiments with learners consistently suggest that students with more (spontaneous) self-explanation learn and remember better [5, 19, 17, 3] and that a stimulation of self-explanation leads to better learning results [22, 4, 18]. ‘Erroneous examples’ are exercises that present erroneous worked-out examples and ask the learner to find-and-correct or to correct the errors.

Now, self-explanation can also be fostered when requested for erroneous worked-out examples. Siegler even found a positive effect in favour of explaining both, correctly worked examples and incorrectly worked examples, over simply explaining correctly worked examples [21]. In addition, erroneous examples offer additional opportunities for deep learning as we will see in §3.

As illustrating examples for worked-out examples we have chosen two instances from the (mathematical) area convergence of sequences which is taught in high school and university. This choice does not introduce particular restrictions but implies multi-step problem solving and may exhibit a variety of error types which we discuss below.

Worked Example 1. We want to prove: If $a > 0$, then the sequence $\left(\frac{1}{1+n \cdot a}\right)$ converges to 0.

Since $a > 0$ it follows that $0 < n \cdot a < 1 + n \cdot a$. Therefore, we conclude that

$$0 < \frac{1}{1+n \cdot a} < \frac{1}{n \cdot a}, \text{ which evidently implies that } \left| \frac{1}{1+n \cdot a} - 0 \right| = \frac{1}{a} \cdot \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Since $\left(\frac{1}{n}\right)$ converges to 0, we may invoke Theorem... with $c = \frac{1}{a}$ and $m = 1$ to infer that

$\left(\frac{1}{1+n \cdot a}\right)$ converges to 0. QED.

An erroneous example can be generated in which the (underlined) condition is eliminated. If the condition ' $a > 0$ ' in the conjecture and correspondingly the ' $a > 0$ it follows that' (both underlined) in a proof step are removed, then this is not a valid proof anymore. Such missing conditions and omitted restrictions are a frequent mistake in mathematical problem solving which make a solution or proof incorrect (e.g., through a division by 0).

Worked Example 2 We want to show: The sequence $\left((-1)^n\right)$ is divergent.

Assume that a limit a exists (assumption of proof by refutation).

Then for each ε there is a k such that for all $n > k$ holds $\left|(-1)^n - a\right| < \varepsilon$.

Let $\varepsilon = 1$, then there is a k such that for all $n > k$ holds $\left|(-1)^n - a\right| < 1$.

If n is odd and $n > k$, it gives $|-1 - a| < 1$ so that $-2 < a < 0$.

If n is even and $n > k$, it gives $|1 - a| < 1$ so that $0 < a < 2$.

Since a cannot satisfy both inequalities the assumption of convergence leads to a contradiction. Therefore, this assumption is false and this implies $\left((-1)^n\right)$ is divergent. QED.

From this worked-out proof an erroneous example can be generated by removing the (underlined) case 'n is even'. This illustrates a typical error of a problem solution in which one or more cases resulting from a case split (in this example (n is even) or (n is odd)) are not considered. The error would be relatively simple to detect, if only the underlined part is omitted, the later phrase 'both inequalities' still points to the missing part – at least if properly interpreted. It may be much harder, if the case split is missing altogether.

As clear not just from these examples, there are different types of errors which also require different skills and efforts for detection and correction. For instance, some errors can be found by inspecting the single steps in a worked-out example. Presumably, for most students those errors are easier to detect than conceptual misconceptions or mistakes in the overall structure of a solution. Some errors require to analyze the problem solution and to reason why this cannot be correct as a whole. These might require meta-cognitive activities.

3. ERRONEOUS EXAMPLES AS A SOURCE OF LEARNING

Pedagogic investigations [14] suggest that a better error-culture in schools will improve several conditions for learning and the performance of the students. For instance, it will reduce anxieties, improve intrinsic motivation, improve the self-effectiveness of the learner and the internal attribution of success and failure.

3.1 What can be stimulated by making use of errors?

Cognitive investigations and our own experiences suggest that a proper usage of erroneous examples will stimulate meta-cognition such as self-explanation, reflection, inquiry, critical thinking, diverse reasoning strategies and epistemological attitudes. More specifically,

- The task to search, explain, and correct errors in (somebody else's) solution can foster self-explanation

- Spontaneously recognizing an error implies that a result or situation does not meet one's expectation or does not make sense and therefore this situation can be a stimulus for reflection and exploration.
- Errors in worked examples can serve as a source for stimulating inquiry learning since the learner has to explore the potential problem solving space, think about alternatives, discover the structure, dependencies, and essence of a worked example. At the same time, the worked example provides a skeleton for the learner's exploration.
- Working on erroneous examples and other forms of using failures productively can help in practicing critical thinking.
- Since the goal of working on erroneous example exercises is different from the usual goal in problem solving exercises the focus of the learner shifts from performance-orientation towards learning-orientation. This requires carefully chosen prompts and giving credit for analyzing a solution rather than applying (memorized) rules. This goal shift is similar for self-explanation exercises which proved useful for the learning (and the learning motivation). We believe that this aspect is extremely important for the impact of self-explanation and of erroneous examples.
- Handling of mistakes is a decisive part of meta-cognition. More specifically, finding an error in an erroneous example could be useful for learning because the student has to look at alternative (including erroneous) ways to solve a problem. However, the memorization of errors instead of correct steps has to be prevented. Similarly, error identification may help to develop self-monitoring for situations, where similar errors are made by the student herself
- The student may train to reason (backwards) from a solution and to find inferences/rules that led to this solution.
- Debugging an erroneous example is somewhat similar to the natural role (of a learner) as tutor for other learners which is an effective learning strategy itself.
- As a new type of exercises, trying to find the mistakes can motivate learners and can be fun. Similarly, working like a detective might be motivating and focusing the learner's attention
- Finding errors that are frequent in real life (e.g., wrong results when working with a calculator or confusion of dimensions in physics solutions) will help learners to better master real life demands.

The emotional factor, in particular anxiety in learning, seems to play an enormous role in dealing with errors and it influences the overall learning in a negative way [7]. This applies especially, when the student makes a mistake herself rather than when discussing errors of others (maybe even distant persons). Therefore, the task to analyze somebody else's erroneous solution in an erroneous example may be less critical emotionally than analyzing my own solution.

For those and other reasons, a proper usage of erroneous examples means a usage adapted to the learner's needs, capabilities, and context such that the learner is able to discover and correct the errors (which depends on previous training, on mastery, on previously seen examples) and that the learner's self-image is not violated.

3.2. Types of Errors

For the design of erroneous examples for a learning environment we have to investigate the types of errors that are frequent for problems in a particular mathematical domain in the first place. We did this for the class of proofs presented above, for derivative problems and for examples for computation with fractions. The frequent errors in the proofs include

- Missing or erroneous conditions
- Misconception of quantifiers
- Missing case splits or missing subproofs (taken for granted)
- Erroneous variable handling
- Misconception of proof by refutation (strategic error).

For the domain 'derivation of (composite) functions' the types of errors are different. They include

- Misconception of the interpretation of the notion 'derivative'

- Misconception of composite functions
- Erroneous assumptions about composite functions
- Application of wrong derivation rules or wrong application of such rules
- Misconception about variables
- Missing domain conditions
- Slips in computations
- Arithmetic or algebraic errors .

4. INSTRUCTIONAL STRATEGIES CAPITALIZING ON ERRORS

Pedagogical strategies dealing with errors vary from avoiding errors at the one extreme to inquiry learning stimulated by errors on the other extreme. A strategy initiating inquiry learning through error analysis by learners [1] makes the following epistemological (and other) assumptions

(1) Mathematics is not absolute and immutable but fallible, informed by purposes, context, etc. Critical thinking is, thus, appropriate. (2) The mathematical knowledge is not a stable body of established results but results from a dynamic process of inquiry, where conflict, doubt and search for refined understanding of the world is normal. (3) Teaching is viewed as creating a rich environment that stimulate inquiries and support student's own search.

In inquiry learning, students could benefit from capitalizing on errors by raising and pursuing questions such as

- what are the causes and consequences of this error?
- in what circumstances could this result be considered correct?
- what would happen, if we chose to accept this result?

Another tutorial strategy that is easier to implement in an ITS would contrast correctly worked examples with incorrect ones and have the student explain the differences. Yet another strategy asks the student to detect and correct errors in an erroneous example which may involve tutorial questions such as

Is the answer correct or incorrect? Is a principle violated? A rule misused? And, if yes, which one?

5. KNOWLEDGE REPRESENTATION

In order to present erroneous examples as interactive exercises in an intelligent tutoring system that can provide generic and/or specific feedback to a learner, the structural occurrences of a potential error in worked- out solutions need to be determined and represented as entities that can be addressed and replaced. In addition, these entities can be annotated by metadata (e.g., the relation between correctly worked example and erroneous example and type of error).

As a result, at least one correct alternative of an erroneous part is available in the correctly worked example that is stored in the knowledge base and that is linked to the erroneous example by the relations 'erroneous- example for/of' which belong to the metadata. This way, feedback can be generated and the correctly worked example can be for offering help and for generating correction menus.

The erroneous parts have to be annotated with the learning goals that correcting the error may imply.

Learning goals (concepts, methods) and competencies (computation, mathematical argumentation, meta- reasoning, translating to maths language) are metadata that belong to the **ActiveMath** DTD already [2]. These annotation carry the information which enables the course generator to choose erroneous examples useful for a particular student with particular skills and needs.

So far, we identified the following relevant parts of worked examples in **ActiveMath**'s semantic XML knowledge representation, OMDoc [6].

- overall problem statement

- formal expression in the statement
- conditions in problem statement and related condition occurrences in solution
- overall worked solution
- a step in solution
- subsolution (e.g., a case of a case split)
- an assumption of a solution
- formal expression in the solution
- references to arguments established outside the solution
- title of (intermediate) meta- cognitive activities (Polya's heuristics, see [15, 9]).

These parts are represented in an extended OMDoc representation described in [8].

6. HOW ACTIVEMATH CAN MAKE USE OF ERRONEOUS EXAMPLES

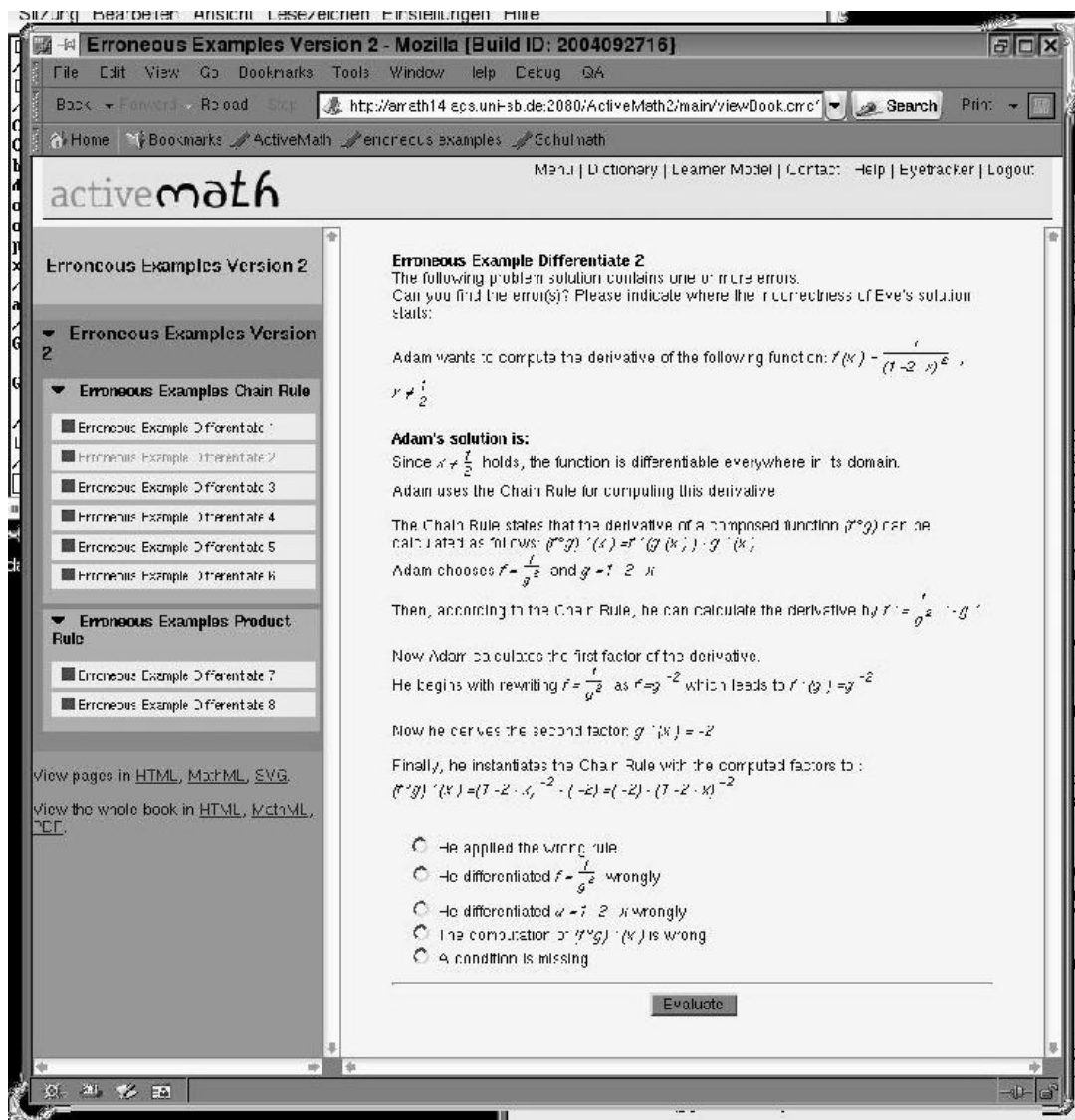
ActiveMath is a user- adaptive learning environment . That is, for course generation **ActiveMath** adapts the content, the sequencing of learning objects, the suggestions to the student to her learning goals, preferences, learning scenarios, context, mastery, etc. For a scenario that presents examples, erroneous examples and exercises for a concept to be learned, it can choose the examples and erroneous examples adaptively. For instance, it can choose an example according to the difficulty that suits the mastery level of the learner for the concept the example illustrates. (For instance, the addition of halves and quarters (which are frequent in real life) is usually simpler than the addition of fractions with denominators 5 or 7. The first would be chosen for a student who is still weak in adding fractions.)

ActiveMath also provides so- called pre- recorded courses on demand of teachers. For these, only the suggestions are provided adaptively. To start with and for usage in a school test, we included erroneous examples into a pre- recorded course. With more empirical evidence, the next step is to suggest erroneous examples adaptively rather than as elements of a fixed course. Then, the course generation will take into consideration previous misconceptions, the learning goal, the capabilities of the learner in order to choose and present erroneous examples and alternate them with examples and exercises.

The presentation of erroneous examples is generated dynamically depending on the learner's capability: erroneous parts are marked for less capable students and only (scaffolded) correction is requested as opposed to stronger learners who are asked to find and correct errors. As a guideline, heuristics and methods on how to find errors have to be taught before a student is confronted with erroneous examples. Otherwise it may be difficult for many students to perform the still unusual task of correcting errors.

Currently, the system employs multiple choice questions for localizing the error as shown in Figure 1. Soon this will be replaced by the (more difficult) request to mark the erroneous parts. Currently, the feedback addresses correctness/incorrectness and can give away the correct solution. More intelligent feedback has still to be authored. We plan, however, to generate some feedback from the classification of errors and other metadata. Feedback is crucial, especially for weak learners and it reduces the time spent on a problem. Finding errors might have a stronger effect when combined with prompts for explaining the internal reasoning and thereby making it explicit. Thus, we will design prompts requesting explaining the actual error as well as explaining why the overall solution is wrong.

Figure 1: Screenshot from **ActiveMath** with a pre- recorded course with erroneous examples



7. CONCLUSION

We discussed the features and design of erroneous examples for learning of mathematics in general and more specifically learning with the web-based learning environment **ActiveMath**.

When erroneous examples are delivered in **ActiveMath**, the main objectives are to improve meta-reasoning (e.g. self-monitoring) and critical thinking as well as deep learning and knowledge application in real life (e.g., a teacher needs to be able to find errors in students' solutions, an engineer needs to find errors in a machine process).

Moreover, research and empirical investigations have to be conducted to answer difficult questions such as:

- how to devise most effective feedback in erroneous examples exercises?
- when is it appropriate to present erroneous examples to a student? And how to present them adaptively?
- which combination of examples, erroneous examples and exercises works best for which type of student?

- what is the effect of negative feedback?
- under which conditions are errors memorized as correct rather than as errors and how to prevent a student from memorizing errors ?

Many teachers believe that erroneous examples can be harmful because some students might memorize an error without the error- connotation and thereby the error will be duplicated more often rather than eliminated from reasoning. This folklore has to be questioned empirically, and special care has to be taken to avoid such an effect.

8. ACKNOWLEDGMENT

This article was written in the context of the LeActiveMath project, funded under the 6th Framework Program of the European Community – (Contract No IST-2003- 507826). The author is solely responsible for its content.

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